Simultaneous Inversion of Event m_{Lg} and Path Attenuation Coefficient with Application to a Transportable L_g Magnitude Scaling[1]

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Abstract

A simple, consistent, and transportable magnitude scale for regional phases is useful and often required in order to (A) improve the source discrimination capability, (B) determine the station detection threshold, and (C) estimate the explosive yield. A convenient, and hence recommended, magnitude formula for L_g phase is:[2]

$$m_{L_g} \equiv 4.0272 - \text{Bias} + \log A(\Delta) + \frac{1}{3}\log(\Delta(\text{km})) + \frac{1}{2}\log[\sin(\frac{\Delta(\text{km})}{111.1(\text{km/deg})})] + \frac{\gamma(\Delta - 10\text{km})}{\ln(10)}, \qquad [1]$$

where the "bias" term is meant to account for the different L_g excitation relative to m_b. For instance, a bias of approximately 0.39 magnitude unit for Iranian Plateau has been suggested by Nuttli.[3]

Given a suite of events with L_g phases recorded at a seismic network, we present an iterative procedure to simultaneously invert for the path γ and the event m_{L_g} values in Equation [1] without using *a priori* path γ information. Independently derived γ (or Q) values, if available, can be utilized to further constrain the trade-off between the bias term in [1] and the resulting γ values. Other constraints can be easily incorporated into this iterative inversion scheme as well. The procedure is less sensitive to rounding errors, and hence it is numerically more accurate than those direct methods based on matrix factorization or Gaussian elimination. When the number of equations becomes large, the iterative approach is often the only practical means to tackle the inversion. This joint inversion scheme is a natural extension to crustal phases of the one we previously used in the teleseismic analyses.[4]

The proposed joint inversion scheme has been tested with Pahute Mesa and Novaya Zemlya explosions, and the $m_b - L_g$ bias at these two sites are inferred to be −0.34 and −0.26 magnitude unit, respectively. These bias estimates are "optimal" in that the resulting path Q_0 values, which are the by-product of this inversion exercise, would be in best agreement with those derived by the coda-Q method.[5] [6] [7] This exercise yields twenty and eleven stations calibrated for L_g phase from Pahute Mesa and Novaya Zemlya regions, respectively.

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[2] Jih, R.-S. and C. S. Lynnes (1993). Studies of regional phase propagation in Eurasia. *Report PL-TR-93-2003 (=TGAL-93-01),* Phillips Laboratory, Hanscom AFB, MA (ADA262801).

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Project Objective

Develop simple, transportable magnitude scales for miscellaneous regional phases. Review and improve inversion techniques currently in use. Some anticipated products/payoffs include:
[1] a more consistent way of calibrating a suite of propagation paths,
[2] refined event magnitudes which have immediate or potential use in (a) source discrimination study, (b) determination of station detection threshold, and (c) yield estimation.

Research Accomplished

To date the absolute L_g magnitude scale is defined on a region-by-region basis. For Eastern U.S., Nuttli's (1986ab) two-step formulae are equivalent to the following one-step procedure (Jih and Lynnes, 1993):

$$m_{L_g} \equiv 4.0272 + \log A(\Delta) + \frac{1}{3}\log(\Delta(\text{km})) + \frac{1}{2}\log[\sin(\frac{\Delta(\text{km})}{111.1(\text{km/deg})})] + \frac{\gamma(\Delta-10\text{km})}{\ln(10)} . \quad [1]$$

This formula defines a magnitude scale such that a seismic source with 1-sec L_g amplitude of 110 μm at 10 km (extrapolated) epicentral distance would correspond to a m_{L_g} of 4.0272 + 2.0414 + 0.3333 - 1.4019 + 0.0000 = 5.000, which was suggested to be appropriate for both eastern North America and Semipalatinsk. That is to say, a seismic source in these two regions with m_b 5.0 would have a m_{L_g} approximately the same.

For Iranian Plateau, Nuttli (1980) reported that seismic sources with the ISC bulletin m_b 5.0 excite L_g amplitudes approximately 270 microns at a 10-km extrapolated distance. If m_{L_g} scale is to be "normalized" to m_b scale at m_b = 5.0, then Equation [1] would have to be revised for Iran as:

$$m_{L_g} \equiv 4.0272 - \text{Bias} + \log A(\Delta) + \frac{1}{3}\log(\Delta(\text{km})) + \frac{1}{2}\log[\sin(\frac{\Delta(\text{km})}{111.1(\text{km/deg})})] + \frac{\gamma(\Delta-10\text{km})}{\ln(10)} , \quad [2]$$

where a "bias" of approximately 0.39 magnitude unit [m.u.] is added to account for the different L_g excitation (relative to m_b) observed in Iranian Plateau. Two fundamental issues arise immediately:

[A] If we accept Equation [2] as the general definition of m_{L_g} scale, what is the "bias" term appropriate for other places, say western North America (such as NTS) or Novaya Zemlya regions?

[B] The path anelastic attenuation coefficient, γ, is assumed to be known before Equation [1] (or [2]) can be applied. How should the path attenuation coefficient, γ, be determined for regions like Novaya Zemlya where L_g might be blocked rather than absorbed through the intrinsic attenuation? Would the coda-*Q* method still be applicable to those blocked paths?

Using L_g amplitudes collected under a recently completed AFTAC contract F08606-91-C-0005 (Baumstark and Wagner, 1994), these two issues are partially examined with an inversion algorithm which simultaneously determines the event sizes, the bias term, and the path corrections without utilizing any *a priori* path γ information. The original formulation of this iterative inversion procedure was first presented in Jih (1992), and it was tested with some quasi-synthetic data. Its updated version is briefly described in the Appendix below.

Given a postulated bias value, a system of linear equations (based on Equation [2]) can be solved for the event m_{L_g} and path γ values. If the path attenuation coefficients are readily available from other studies independently, then such extra information can be used to further constrain the bias term for the

most probable solution. This is exactly the approach to be used in the following exercises.

1. Novaya Zemlya Results

For Novaya Zemlya test site, there are 24 explosions recorded at 13 stations, totaling 92 L_g paths. The average m_G of these 24 events is 5.88, based on the path-corrected / station-corrected m_G reported in Jih and Baumstark (1994) (*cf.* pages 27-28). This value was used to constrain the inversion. The Q_0 and η values associated with each of five postulated bias terms ranging from 0.0 to 0.40 m.u. are listed in Table 1. It turns out that adopting a bias of 0.26 m.u. in Equation [2] would lead to the best agreement between the resulting Q_0 values and those measured by Nuttli (1988) with the coda-Q method.

Table 1. Q_0, η of Novaya Zemlya – WWSSN Paths

Station	Nuttli	Postulated L_g-m_b Bias and Resulting Q_0				
Code	BSSA 1988	.00	.10	.20	.26*	.40
COP	633 0.4	802 0.86	745 0.87	697 0.89	670 0.89	615 0.91
DAG	___ ___	290 0.68	279 0.69	268 0.71	262 0.71	249 0.73
ESK	___ ___	499 0.67	481 0.69	463 0.70	454 0.71	433 0.73
IST	___ ___	592 0.71	569 0.72	547 0.73	536 0.74	509 0.75
KBS	315 0.5	___ ___	___ ___	___ ___	___ ___	___ ___
KEV	252 0.6	314 0.50	292 0.52	274 0.54	263 0.55	242 0.57
KON	496 0.5	___ ___	454 0.04	433 0.10	421 0.13	396 0.20
NOR	___ ___	243 0.47	235 0.49	228 0.51	223 0.53	213 0.55
NUR	420 0.5	512 0.64	479 0.66	450 0.67	434 0.68	401 0.70
STU	531 0.5	603 0.62	577 0.64	553 0.65	540 0.66	512 0.68
TRI	___ ___	521 0.48	502 0.51	485 0.52	475 0.54	454 0.56
UME	391 0.5	456 0.87	427 0.88	401 0.89	386 0.89	358 0.90

2. Pahute Mesa Results

For Pahute Mesa explosions, a L_g-m_b bias of 0.34 m.u. (Table 2) appears to give Q_0 values most consistent with those Nuttli (1986a) and Patton (1988) obtained. 225 L_g signals recorded at 21 Eurasian stations were used in this inversion. The 47 Pahute Mesa events have an average m_G of 5.51 (*cf.* pages 17-18 of Jih and Baumstark, 1994).

Table 2. Q_0, η of Pahute Mesa – WWSSN Paths						
Station	Nuttli+Patton	Postulated $L_g - m_b$ Bias and Resulting Q_0				
Code	BSSA 86, 88	.00	.10	.20	.34*	.40
AAM	— —	904 1.11	838 1.11	782 1.10	714 1.10	689 1.10
ALQ	— —	264 0.86	247 0.87	231 0.88	212 0.89	205 0.90
ATL	— —	— —	360 0.01	352 0.07	340 0.14	336 0.17
BKS	139 0.6	180 0.88	166 0.89	153 0.89	139 0.90	133 0.90
BLA	— —	560 0.38	537 0.42	515 0.45	488 0.48	476 0.50
CMB	— —	138 0.36	125 0.43	115 0.49	102 0.56	98 0.59
COR	— —	203 0.26	193 0.30	184 0.35	173 0.39	168 0.41
ELK	150 0.5	287 0.47	243 0.52	211 0.56	179 0.60	168 0.61
FVM	— —	373 0.12	359 0.17	346 0.21	328 0.26	321 0.28
GOL	— —	234 0.53	221 0.56	209 0.58	195 0.61	190 0.62
JAS	— —	149 0.66	134 0.69	123 0.71	109 0.74	104 0.75
JCT	— —	456 0.72	428 0.74	402 0.76	371 0.78	359 0.79
KNB	142 0.4	218 0.79	181 0.81	157 0.82	132 0.82	124 0.83
LAC	097 0.7	189 0.53	164 0.59	145 0.65	124 0.71	117 0.73
LON	— —	202 0.58	194 0.60	186 0.62	176 0.65	172 0.66
LUB	— —	423 1.57	393 1.55	367 1.52	337 1.49	325 1.48
MNV	093 0.6	— —	— —	140 0.24	110 0.45	100 0.52
OGD	— —	664 0.48	637 0.52	611 0.55	577 0.58	564 0.59
SCP	— —	520 0.04	514 0.13	506 0.20	491 0.29	484 0.32
WES	— —	1205 1.02	1114 1.01	1035 1.01	942 1.00	908 1.00

Conclusions and Recommendations

For Pahute Mesa, where the Q_0 (and η) values derived by the coda-Q method are believed to be appropriate to account for the path attenuation, the $L_g - m_b$ bias is estimated as 0.34 m.u. This value happens to be in agreement with the published m_b bias caused by the different upper mantle absorption in the eastern and western U.S. Thus this Pahute Mesa exercise might be in support of Nuttli's assertion that the same absolute L_g magnitude scale can be used for both eastern and western U.S. However, this is probably an exception rather than a general rule.

Previously only a handful number of stations was analyzed by Nuttli (1986a) and Patton (1988) with the coda-Q method. We now have 20 "calibrated" stations for L_g phases from Pahute Mesa. However, the proposed calibration is subject to the choice of the "bias" parameter which is not quite obvious to decide. This indeterminacy of L_g absolute magnitude scale could be a persistent issue encountered in every attempt of using seismic phases like L_g which is not as transportable as M_S. A (new) feature of this simultaneous inversion code is that it offers a suite of solutions to choose from. If only a "relative" L_g scale is of interest, then setting the bias to an arbitrary value, say 0, would suffice. In any case, the simultaneous inversion can calibrate a suite of stations in a more consistent manner, which is the typical advantage of GLM inversion schemes.

The same procedure gives a $L_g - m_b$ bias of 0.26 m.u. for northern Novaya Zemlya explosions. Since it is known that L_g blockage does occur for paths crossing the Barents Shelf, there remains a question whether it is appropriate to use Nuttli's Q_0 values to constrain our selection of the bias term.

Based on the t^* study, it has been suggested that the upper mantle of Novaya Zemlya is similar to that of Semipalatinsk. Thus 0.26 m.u. is expected to be the "upper bound" of the bias term. It seems that, however, the Q_0 values based on the coda-Q method can provide a Q map which is qualitatively consistent with that based on the time-domain computation. The L_g blockage at Barents Shelf could have caused the predominant frequency of L_g waves to shift in a way very similar to that due to a stronger anelastic attenuation. Consequently, Nuttli's Q_0 values may be biased low, yielding an over-compensation of the attenuation effect in computing his Novaya Zemlya m_{L_g} . This conjecture can be tested with numerical modeling experiments using LFD method (*e.g.*, Jih, 1994).

If many stations are deployed along the same azimuth from the shot, *i.e.*, if a profile is available, then the coda-Q results can be verified independently using the inter-station spectral ratios. This could be included in future field experiments.

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Appendix A. Joint Inversion Method: Basic Concepts

Consider the case of m explosions recorded at some or all of n stations. For simplicity, we assume that these m explosions are detonated in the same test site for the moment. The case of multiple test sites will be discussed later. The linear model for L_g phases can then be written as

$$E(i) - \gamma(j)(\Delta(i,j) - 10\text{km})/\ln(10) + \varepsilon(i,j) = Y(i,j)\ , \qquad [3]$$

$$\text{where} \quad Y(i,j) \equiv 4.0272 - \text{Bias} + \log A(i,j) + \frac{1}{3}\log(\Delta(i,j)) + \frac{1}{2}\log[\sin(\frac{\Delta(i,j)}{111.1(\text{km/deg})})]\ .$$

Once the amplitudes and the locations of the events (and hence the epicentral distances, Δ) are available, Y would be completely known. Only the event sizes (E) and the path-specific coefficients of anelastic attenuation (γ) are the unknown parameters to be determined. The obscuring errors ε are assumed to be uncorrelated and to belong to the same probability distribution, namely a common Gaussian distribution with zero mean and variance σ^2.

If all the events are clustered in a small region, then the epicentral distances $\Delta(i,j)$ would be almost identical for a given station j. In this case, our model (Equation [3]) is a special case of a more general linear system:

$$E(i) + S(j) + \varepsilon(i,j) = Y(i,j)\ , \quad \text{for } i = 1,\dots, m\ ; j = 1,\dots, n\ . \qquad [4]$$

which can be expressed in a matrix formulation:

$$\mathbf{H}\,X + \mathbf{e} = Y\ , \qquad [5]$$

where **H** is the design (or observation) matrix. X and Y are the column vectors of unknowns and observations, respectively. The standard least-squares [LS] solution (viz, the one that minimizes the *residual sum of squares:* RSS $\equiv (Y - \mathbf{H}\hat{X})' (Y - \mathbf{H}\hat{X})$) to *any* linear system with a general form like Equation [5] is

$$\hat{X}_{LS} \equiv (\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'Y\ , \qquad [6]$$

where $\mathbf{H}'$ is the transpose of **H**. This least-squares estimator has many optimality properties. For instance, it is unbiased, and it gives minimum variance within the class of linear unbiased estimators. Furthermore, $\hat{X}_{LS}$ is also the *Maximum-Likelihood Estimate* [MLE] under the Gaussian assumption. It is straightforward to compute the uncertainty by using Var$[\hat{X}_{LS}] \equiv$ the diagonal of $\sigma^2(\mathbf{H}'\mathbf{H})^{-1}$, which is simply

scaling the variance of the random perturbations by the number of observations associated with each unknown.

In our case, however, the matrix **H′H** in Equation [6] is singular, and hence the least-squares theory can not be applied immediately unless the linear system of [4] is modified somewhat. Perhaps the easiest way to illustrate the indeterminacy due to the singularity of the matrix **H′H** is that given any set of solution to [4], we can always obtain yet another set of solution by adding a constant to all event magnitudes, E(i), i = 1,..., m, and subtracting the same constant from each station term S(j) (Jih and Shumway, 1989). Alternatively, we can verify that the matrix **H′H** has zero determinant with linear algebra packages such as LINPACK, EISPACK, or LAPACK. In this study, however, a formal proof is presented below. Without loss of generality, we can assume that each of the m events is fully recorded at all n stations, then

$$X \equiv [E_1, E_2, E_3, \ldots, E_m, S_1, S_2, \ldots, S_n]', \quad \mathbf{H'H} = \begin{bmatrix} n\cdot I_m & 1_{mxn} \\ 1_{nxm} & m\cdot I_n \end{bmatrix}$$

where I_m is the identity matrix of order m, and all elements of the m-by-n matrix 1_{mxn} are 1. For instance, if m = 3 and n = 2, then

$$\mathbf{H'H} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 & 3 \end{bmatrix}$$

which is a *doubly-bordered band diagonal* sparse matrix (Tewarson, 1973; Press *et al.*, 1988). After exactly n row operations eliminating the lower-left submatrix, 1_{nxm}, the determinant of **H′H** can be computed (up to a multiplicative constant) as that of

$$\begin{bmatrix} I_m & (\frac{1}{n})_{nxm} \\ 0_{nxm} & P_n \end{bmatrix} \qquad [7]$$

where P_n is a square matrix of order n with $\frac{-m}{n}(n-1)$ on the diagonal and $\frac{-m}{n}$ elsewhere. For m = 3 and n = 2, [7] becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1.5 & -1.5 \\ 0 & 0 & 0 & -1.5 & 1.5 \end{bmatrix}$$

It suffices to examine the determinant of this P_n, or, equivalently, to check the determinant of a square matrix of order n with n-1 on the diagonal and -1 elsewhere:

$$\begin{bmatrix} n-1 & -1 & -1 & \cdot & -1 \\ -1 & n-1 & -1 & \cdot & -1 \\ -1 & -1 & n-1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & -1 & -1 & \cdot & n-1 \end{bmatrix}$$

It is straightforward to prove that, by *mathematical induction,* this matrix has zero determinant for any n ≥ 2. Thus the matrix **H′H** in our linear model is always singular regardless how good the observed amplitudes/magnitudes, Y, might be. We therefore need an extra boundary condition to constrain our linear model for a unique solution. The most commonly adopted approach in teleseismic magnitude

determination is to have all station terms sum up to zero. This implies that larger events would tend to be unchanged whether we apply the station corrections or not, and hence the bulletin magnitudes of larger events published by ISC and NEIC can be regarded as more or less unbiased. This extra constraint can be incorporated into [4] by replacing all S(n) by $-\sum_{j=1}^{n-1} S(j)$. It not only reduces the number of unknowns by one, but, more importantly, regularizes the whole linear system to make **H'H** invertible. However, this is not the only plausible constraint. We can impose the extra constraint on E(i) instead, or, even impose the constraint on *some* selected stations.

A first glance of Equation [4] might lead to a conclusion that the inversion scheme for crustal phases is identical to that for teleseismic phase, and hence the algorithms and the constraints suitable for the teleseismic data reduction would be appropriate for the regional case as well. This is not the case. There are generally fewer regional stations available for inversions with crustal phases. The implicit assumption that the recording stations are evenly (and randomly) distributed may not be valid. Some S(j) terms may carry more weight due to the larger corresponding $\Delta(\cdot,j)$. This implies that the zero-sum assumption on the S terms may not be appropriate for the L_g inversion. In fact, the S term in Equation [4], by definition, can not have zero sum across the network because both γ and Δ in Equation [3] are always non-negative.

Appendix B. Joint Inversion Method: The Iterative Procedure

Once a constraint has been chosen, the inverse matrix of **H'H** in Equation [6] can be computed with matrix factorization (such as *Singular Value Decomposition,* [SVD]) or *Gaussian elimination* method. Numerical algorithms of these types are called *direct methods.* Direct methods can be impractical if **H'H** is large. In that case, iterative methods are often the only possible method of solution, as well as being faster and more accurate than Gaussian elimination and matrix factorization. The largest area for the application of iterative methods is that of the linear systems arising in the numerical solution of partial differential equations. Systems of orders 10,000 to 100,000 are not unusual in aerospace sciences, although the majority of the coefficients of the systems are typically zeros.

The basic idea of iterative methods is that one starts with a trial solution vector $X^{(0)}$ and carries out some process using **H**, Y, and $X^{(0)}$ to get a new vector $X^{(1)}$. Then one repeats. At the k stage, one uses the iterative process to get $X^{(k)}$ from **H** (or **H'H**), Y, and $X^{(k-1)}$. The specific algorithm for our problem is summarized in five steps:

Step 0

Set initial value of $\gamma(j)$ for j = 1,..., n.

Step 1

Compute event m_{L_g} , E(i), for i = 1,..., m:

$$E(i) = \frac{1}{\#(j)}\sum_j [Y(i,j) + \gamma(j)[\Delta(i,j)-10\text{km}]/\ln(10)] ,$$

where #(j) is the number of stations used in the summation.

Step 2

Adjust E(i), i = 1,..., m, with the desired boundary condition.

Step 3

Compute the path-specific coefficient of anelastic attenuation, $\gamma(j)$, for j=1,..., n:

$$\gamma(j) = \ln(10)\sum_i [\ E(i) - Y(i,j)\ ]/\sum_i [\Delta(i,j) - 10km]\ .$$

Step 4

Fill in missing paths (i,j) with predicted pseudo-observations:

$$Y(i,j) \equiv E(i) - \gamma(j)(\Delta(i,j) - 10km)/\ln(10)\ .$$

Step 5

Repeat steps [1]-[4] to update E and γ till convergence.

Although an arbitrary guess of γ will do at Step 0, picking an initial γ close to the average across the whole area of interest would speed up the convergence. The choice of the extra boundary condition (at Step 2) is very flexible. Constraining the mean of all event m_{L_g} seems to perform extraordinarily well, however. Note that the pseudo-observations predicted at Step 4 are treated as "good" observations at steps 1 and 3 except during the first iteration loop.

The iterative procedure described above is a special case of a general algorithm known variously as the *Gaussian-Seidel* method, the *Liebmann process,* or the *method of successive displacements* (Bunch and Rose, 1976; Forsythe *et al.*, 1977; Golub and Van Loan, 1983; Spedicato, 1991). The major difference between this method and that of *Gaussian-Jacobi (viz.,* the *simultaneous displacement* method) is that we solve for one component (viz, E(i) or $\gamma(j)$) of the new vector X using for each other component of X its most recently computed value, whereas Gaussian-Jacobi method updates all unknown parameters simultaneously at the end of each iteration loop. In our case, Gaussian-Seidel method converges faster than does Gaussian-Jacobi method. The first application of this iterative technique to the magnitude determination problem was Blandford and Shumway (1982), although the methodology was identified as the E-M [Expectation-Maximization] algorithm (Dempster *et al.*, 1977) following the convention in the statistical community.

The advantage of our multi-event joint inversion scheme, as compared to the simple network averaging for each individual event that Nuttli (1986ab, 1988) used, is that we can have a more consistent network for all events. By "consistent" it means that the joint inversion procedure provides the best approximation to the network-averaged values that would have been obtained if all the events were recorded at every station in the network. This advantage may not be very obvious in using direct methods such as SVD or Gaussian elimination. However, it would become quite natural as revealed by Step 4 of our iteration scheme.